5-7 Analyzing Graphs of Quadratic Functions

Main Ideas

- Analyze quadratic functions of the form $y = a(x - h)^2 + k$.
- Write a quadratic function in the form $y = a(x - h)^2 + k$.

New Vocabulary

vertex form

GET READY for the Lesson

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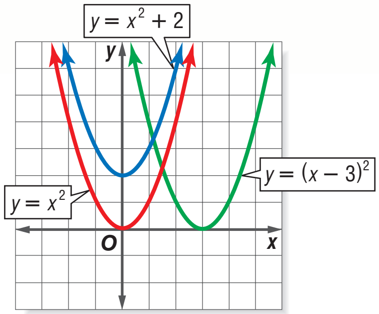
A *family of graphs* is a group of graphs that displays one or more similar characteristics. The graph of $y = x^2$ is called the *parent graph* of the family of quadratic functions.



The graphs of other quadratic functions such as $y = x^2 + 2$ and $y = (x - 3)^2$ can be found by transforming the graph of $y = x^2$.

Analyze Quadratic Functions Each function above can be written in the form $y = (x - h)^2 + k$, where (h, k) is the vertex of the parabola and $x = h$ is its axis of symmetry. This is often referred to as the **vertex form** of a quadratic function.

Equation		Axis of
$y = x^2$ or $y = (x - 0)^2 + 0$	(0, 0)	$x = 0$
$y = x^2 + 2$ or $y = (x - 0)^2 + 2$	(0, 2)	$x = 0$
$y = (x - 3)^2$ or $y = (x - 3)^2 + 0$	(3, 0)	$x = 3$

Recall that a *translation* slides a figure without changing its shape or size. As the values of h and k change, the graph of $y = a(x - h)^2 + k$ is the graph of $y = x^2$ translated:

- $|h|$ units *left* if h is negative or $|h|$ units *right* if h is positive, and
- $|k|$ units *up* if k is positive or $|k|$ units *down* if k is negative.

EXAMPLE Graph a Quadratic Equation in Vertex Form

1 Analyze $y = (x + 2)^2 + 1$. Then draw its graph.

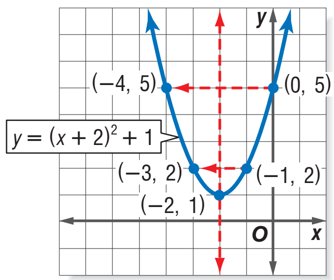
This function can be rewritten as $y = [x - (-2)]^2 + 1$. Then $h = -2$ and $k = 1$. The vertex is at (h, k) or $(-2, 1)$, and the axis of symmetry is $x = -2$. The graph is the graph of $y = x^2$ translated 2 units left and 1 unit up.

Now use this information to draw the graph.

Step 1 Plot the vertex, $(-2, 1)$.

Step 2 Draw the axis of symmetry, $x = -2$.

Step 3 Use symmetry to complete the graph.



CHECK Your Progress

1. Analyze $y = (x - 3)^2 - 2$. Then draw its graph.

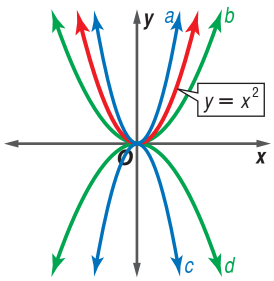
How does the value of a in the general form $y = a(x - h)^2 + k$ affect a parabola? Compare the graphs of the following functions to the parent function, $y = x^2$.

a. $y = 2x^2$

b. $y = \frac{1}{2}x^2$

c. $y = -2x^2$

d. $y = -\frac{1}{2}x^2$



All of the graphs have the vertex (0, 0) and axis of symmetry $x = 0$.

Notice that the graphs of $y = 2x^2$ and $y = \frac{1}{2}x^2$ are *dilations* of the graph of $y = x^2$. The graph of $y = 2x^2$ is narrower than the graph of $y = x^2$, while the graph of $y = \frac{1}{2}x^2$ is wider. The graphs of $y = -2x^2$ and $y = 2x^2$ are *reflections* of each other over the x-axis, as are the graphs of $y = -\frac{1}{2}x^2$ and $y = \frac{1}{2}x^2$.

Changing the value of a in the equation $y = a(x - h)^2 + k$ can affect the direction of the opening and the shape of the graph.

- If $a > 0$, the graph opens up.
- If $a < 0$, the graph opens down.
- If $|a| > 1$, the graph is narrower than the graph of $y = x^2$.
- If $0 < |a| < 1$, the graph is wider than the graph of $y = x^2$.

Study Tip

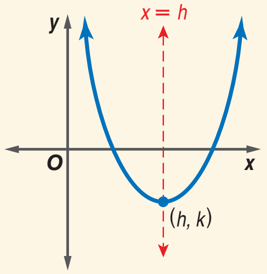
$0 < |a| < 1$ means that a is a real number between 0 and 1, such as $\frac{2}{5}$, or a real number between -1 and 0, such as $-\frac{\sqrt{2}}{2}$.

CONCEPT SUMMARY — *Quadratic Functions in Vertex Form*

The vertex form of a quadratic function is $y = a(x - h)^2 + k$.

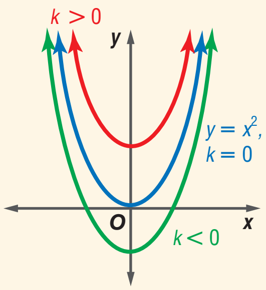
h* and *k

Vertex and Axis of Symmetry



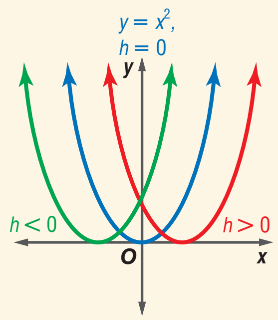
k

Vertical Translation



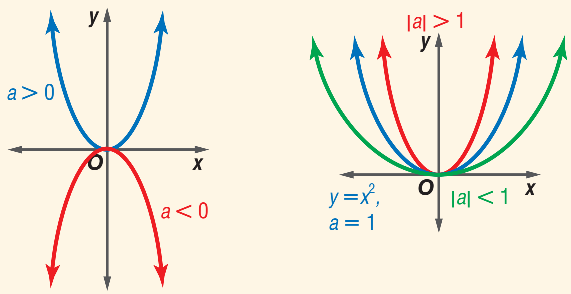
h

Horizontal Translation



a

Direction of Opening and Shape of Parabola



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STANDARDIZED TEST EXAMPLE Vertex Form Parameters

2 Which function has the widest graph?

A $y = -2.5x^2$ **B** $y = -0.3x^2$ **C** $y = 2.5x^2$ **D** $y = 5x^2$

Read the Test Item

You are given four answer choices, each of which is in vertex form.

Solve the Test Item

The value of a determines the width of the graph. Since $|-2.5| = |2.5| > 1$ and $|5| > 1$, choices A, C, and D produce graphs that are narrower than $y = x^2$. Since $|-0.3| < 1$, choice B produces a graph that is wider than $y = x^2$. The answer is B.

Test-Taking Tip

The sign of a in the vertex form does not determine how wide the parabola will be. The sign determines whether the parabola opens up or down. The width is determined by the absolute value of a.

CHECK Your Progress

2. Which function has the narrowest graph?

F $y = -0.1x^2$ **G** $y = x^2$ **H** $y = 0.5x^2$ **J** $y = 2.3x^2$

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Write Quadratic Equations in Vertex Form Given a function of the form $y = ax^2 + bx + c$, you can complete the square to write the function in vertex form. If the coefficient of the quadratic term is not 1, the first step is to factor that coefficient from the quadratic and linear terms.

EXAMPLE Write Equations in Vertex Form

3 **Write each equation in vertex form. Then analyze the function.**

a. $y = x^2 + 8x - 5$

$y = x^2 + 8x - 5$ Notice that $x^2 + 8x - 5$ is not a perfect square.

$y = (x^2 + 8x + 16) - 5 - 16$ Complete the square by adding $\left(\frac{8}{2}\right)^2$ or 16. Balance this addition by subtracting 16.

$y = (x + 4)^2 - 21$ Write $x^2 + 8x + 16$ as a perfect square.

Since $h = -4$ and $k = -21$, the vertex is at $(-4, -21)$ and the axis of symmetry is $x = -4$. Since $a = 1$, the graph opens up and has the same shape as the graph of $y = x^2$, but it is translated 4 units left and 21 units down.

Study Tip

Check

As a check, graph the function in Example 3 to verify the location of its vertex and axis of symmetry.

b. $y = -3x^2 + 6x - 1$

$y = -3x^2 + 6x - 1$ Original equation

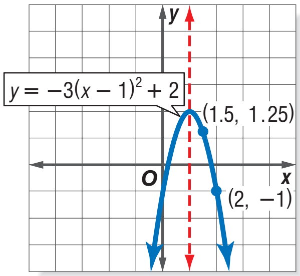
$y = -3(x^2 - 2x) - 1$ Group $ax^2 - bx$ and factor, dividing by a.

$y = -3(x^2 - 2x + 1) - 1 - (-3)(1)$ Complete the square by adding 1 inside the parentheses. Notice that this is an overall addition of $-3(1)$. Balance this addition by subtracting $-3(1)$.

$y = -3(x - 1)^2 + 2$ Write $x^2 - 2x + 1$ as a perfect square.

The vertex is at (1, 2), and the axis of symmetry is $x = 1$. Since $a = -3$, the graph opens downward and is narrower than the graph of $y = x^2$. It is also translated 1 unit right and 2 units up.

Now graph the function. Two points on the graph to the right of $x = 1$ are (1.5, 1.25) and (2, −1). Use symmetry to complete the graph.



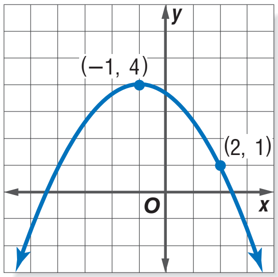
CHECK Your Progress

3A. $y = x^2 + 4x + 6$ **3B.** $y = 2x^2 + 12x + 17$

If the vertex and one other point on the graph of a parabola are known, you can write the equation of the parabola in vertex form.

EXAMPLE 4 Write an Equation Given a Graph

Write an equation for the parabola shown in the graph.



The vertex of the parabola is at (−1, 4), so $h = -1$ and $k = 4$. Since (2, 1) is a point on the graph of the parabola, let $x = 2$ and $y = 1$. Substitute these values into the vertex form of the equation and solve for a.

$y = a(x - h)^2 + k$	Vertex form
$1 = a[2 - (-1)]^2 + 4$	Substitute 1 for y, 2 for x, −1 for h, and 4 for k.
$1 = a(9) + 4$	Simplify.
$-3 = 9a$	Subtract 4 from each side.
$-\frac{1}{3} = a$	Divide each side by 9.

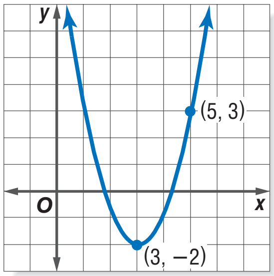
The equation of the parabola in vertex form is $y = -\frac{1}{3}(x + 1)^2 + 4$.

Cross-Curricular Project

Math Online You can use a quadratic function to model the world population. Visit algebra2.com to continue work on your project.

CHECK Your Progress

4. Write an equation for the parabola shown in the graph.



CHECK Your Understanding

Examples 1, 3 (pp. 286, 288)

Graph each function.

1. $y = 3(x + 3)^2$ **2.** $y = \frac{1}{3}(x - 1)^2 + 3$ **3.** $y = -2x^2 + 16x - 31$

Example 2 (p. 288)

4. STANDARDIZED TEST PRACTICE Which function has the widest graph?

A $y = -4x^2$ **B** $y = -1.2x^2$ **C** $y = 3.1x^2$ **D** $y = 11x^2$

Example 3 (pp. 288–289)

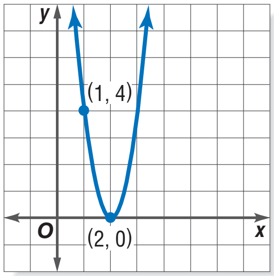
Write each quadratic function in vertex form, if not already in that form. Then identify the vertex, axis of symmetry, and direction of opening.

5. $y = 5(x + 3)^2 - 1$ **6.** $y = x^2 + 8x - 3$ **7.** $y = -3x^2 - 18x + 11$

Example 4 (p. 289)

Write an equation in vertex form for the parabola shown in each graph.

8.



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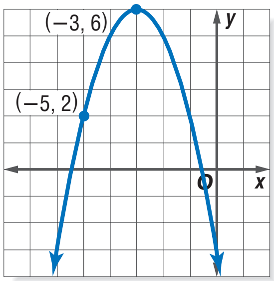

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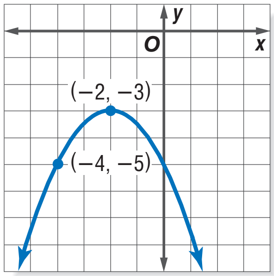
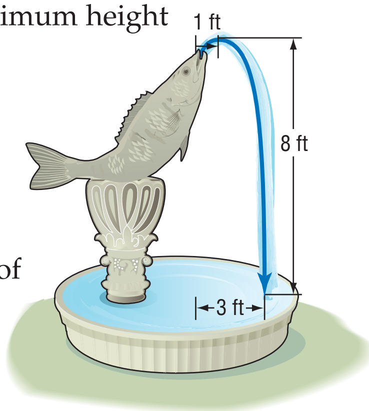


FOUNTAINS The height of a fountain's water stream can be modeled by a quadratic function. Suppose the water from a jet reaches a maximum height of 8 feet at a distance 1 foot away from the jet.



11. If the water lands 3 feet away from the jet, find a quadratic function that models the height $H(d)$ of the water at any given distance d feet from the jet. Then compare the graph of the function to the parent function.

12. Suppose a worker increases the water pressure so that the stream reaches a maximum height of 12.5 feet at a distance of 15 inches from the jet. The water now lands 3.75 feet from the jet. Write a new quadratic function for $H(d)$. How do the changes in h and k affect the shape of the graph?

Exercises

HOMEWORK HELP

For Exercises	See Examples
13–16, 21, 22	1
17–18	1, 3
19, 20	2
23–26, 31, 32	3
27–30	4

Graph each function.

13. $y = 4(x + 3)^2 + 1$ **14.** $y = -(x - 5)^2 - 3$ **15.** $y = \frac{1}{4}(x - 2)^2 + 4$

16. $y = \frac{1}{2}(x - 3)^2 - 5$ **17.** $y = x^2 + 6x + 2$ **18.** $y = x^2 - 8x + 18$

19. What is the effect on the graph of the equation $y = x^2 + 2$ when the equation is changed to $y = x^2 - 5$?

20. What is the effect on the graph of the equation $y = x^2 + 2$ when the equation is changed to $y = 3x^2 - 5$?

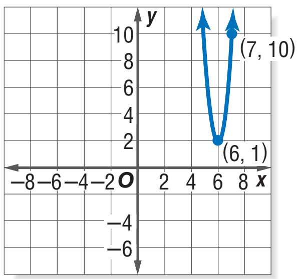
Write each quadratic function in vertex form, if not already in that form. Then identify the vertex, axis of symmetry, and direction of opening.

21. $y = -2(x + 3)^2$ **22.** $y = \frac{1}{3}(x - 1)^2 + 2$ **23.** $y = -x^2 - 4x + 8$

24. $y = x^2 - 6x + 1$ **25.** $y = 5x^2 - 6$ **26.** $y = -8x^2 + 3$

Write an equation in vertex form for the parabola shown in each graph.

27.



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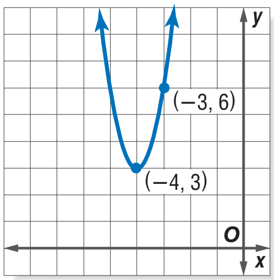

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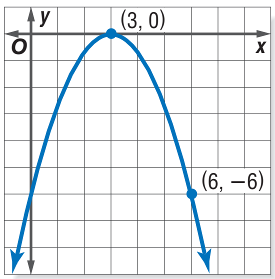
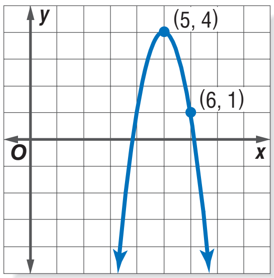


Write an equation in vertex form for the parabola shown in each graph.

30.



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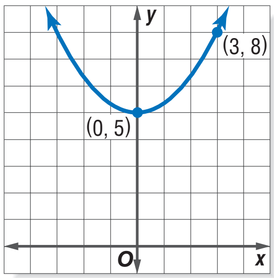

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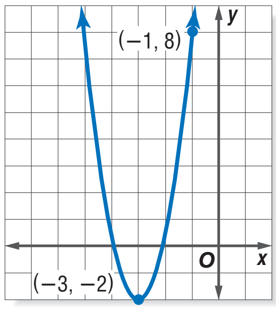

LAWN CARE **For Exercises 33 and 34, use the following information.**
The path of water from a sprinkler can be modeled by a quadratic function. The three functions below model paths for three different angles of the water.

Angle A: $y = -0.28(x - 3.09)^2 + 3.27$
Angle B: $y = -0.14(x - 3.57)^2 + 2.39$
Angle C: $y = -0.09(x - 3.22)^2 + 1.53$

33. Which sprinkler angle will send water the highest? Explain your reasoning.

34. Which sprinkler angle will send water the farthest? Explain your reasoning.

35. Which sprinkler angle will produce the widest path? The narrowest path?

Graph each function.

36. $y = -4x^2 + 16x - 11$

37. $y = -5x^2 - 40x - 80$

38. $y = -\frac{1}{2}x^2 + 5x - \frac{27}{2}$

39. $y = \frac{1}{3}x^2 - 4x + 15$

Write each quadratic function in vertex form, if not already in that form. Then identify the vertex, axis of symmetry, and direction of opening.

40. $y = -3x^2 + 12x$

41. $y = 4x^2 + 24x$

42. $y = 4x^2 + 8x - 3$

43. $y = -2x^2 + 20x - 35$

44. $y = 3x^2 + 3x - 1$

45. $y = 4x^2 - 12x - 11$

46. Write an equation for a parabola with vertex at the origin and that passes through $(2, -8)$.

47. Write an equation for a parabola with vertex at $(-3, -4)$ and y-intercept 8.

48. Write one sentence that compares the graphs of $y = 0.2(x + 3)^2 + 1$ and $y = 0.4(x + 3)^2 + 1$.

49. Compare the graphs of $y = 2(x - 5)^2 + 4$ and $y = 2(x - 4)^2 - 1$.

Real-World Link

The KC135A has the nickname "Vomit Comet." It starts its ascent at 24,000 feet. As it approaches maximum height, the engines are stopped and the aircraft is allowed to free-fall at a determined angle. Zero gravity is achieved for 25 seconds as the plane reaches the top of its flight and begins its descent.

50. **AEROSPACE** NASA's KC135A aircraft flies in parabolic arcs to simulate the weightlessness experienced by astronauts in space. The height h of the aircraft (in feet) t seconds after it begins its parabolic flight can be modeled by the equation $h(t) = -9.09(t - 32.5)^2 + 34{,}000$. What is the maximum height of the aircraft during this maneuver and when does it occur?

DIVING **For Exercises 49–51, use the following information.**
The distance of a diver above the water $d(t)$ (in feet) t seconds after diving off a platform is modeled by the equation $d(t) = -16t^2 + 8t + 30$.

51. Find the time it will take for the diver to hit the water.

52. Write an equation that models the diver's distance above the water if the platform were 20 feet higher.

53. Find the time it would take for the diver to hit the water from this new height.

EXTRA PRACTICE
See pages 901, 930.

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H.O.T. Problems

54. OPEN ENDED Write the equation of a parabola with a vertex of (2, −1) and which opens downward.

55. CHALLENGE Given $y = ax^2 + bx + c$ with $a \neq 0$, derive the equation for the axis of symmetry by completing the square and rewriting the equation in the form $y = a(x - h)^2 + k$.

56. FIND THE ERROR Jenny and Ruben are writing $y = x^2 - 2x + 5$ in vertex form. Who is correct? Explain your reasoning.

Jenny	Ruben
$y = x^2 - 2x + 5$	$y = x^2 - 2x + 5$
$y = (x^2 - 2x + 1) + 5 - 1$	$y = (x^2 - 2x + 1) + 5 + 1$
$y = (x - 1)^2 + 4$	$y = (x - 1)^2 + 6$

57. CHALLENGE Explain how you can find an equation of a parabola using the coordinates of three points on its graph.

58. *Writing in Math* Use the information on page 286 to explain how the graph of $y = x^2$ can be used to graph any quadratic function. Include a description of the effects produced by changing a, h, and k in the equation $y = a(x - h)^2 + k$, and a comparison of the graph of $y = x^2$ and the graph of $y = a(x - h)^2 + k$ using values of your own choosing for a, h, and k.

STANDARDIZED TEST PRACTICE

59. ACT/SAT If $f(x) = x^2 - 5x$ and $f(n) = -4$, which of the following could be n?

A −5

B −4

C −1

D 1

60. REVIEW Which of the following most accurately describes the translation of the graph of $y = (x + 5)^2 - 1$ to the graph of $y = (x - 1)^2 + 3$?

F up 4 and 6 to the right

G up 4 and 1 to the left

H down 1 and 1 to the right

J down 1 and 5 to the left

Spiral Review

Find the value of the discriminant for each quadratic equation. Then describe the number and type of roots for the equation. (Lesson 5-6)

61. $3x^2 - 6x + 2 = 0$ **62.** $4x^2 + 7x = 11$ **63.** $2x^2 - 5x + 6 = 0$

Solve each equation by completing the square. (Lesson 5-5)

64. $x^2 + 10x + 17 = 0$ **65.** $x^2 - 6x + 18 = 0$ **66.** $4x^2 + 8x = 9$

GET READY for the Next Lesson

PREREQUISITE SKILL Determine whether the given value satisfies the inequality. (Lesson 1-6)

67. $-2x^2 + 3 < 0; x = 5$ **68.** $4x^2 + 2x - 3 \geq 0; x = -1$

69. $4x^2 - 4x + 1 \leq 10; x = 2$ **70.** $6x^2 + 3x > 8; x = 0$